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Positive Interval System Dynamics Explored via Vertex Representatives vs. Vertex Majorizations

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***Abstract*—The paper explores the properties of positive interval dynamic systems in continuous-time, via two types of techniques, and develops a comparative study between the two approaches. The first type is based on the properties of the row and column representatives corresponding to the vertex set of the interval matrix $\mathbb{A}=[A^-, A^+]\subset\mathbb{R}^{n\times n}$. The second type employs the properties of the dominant vertex A^+ of the interval matrix. The results, separately derived for the two approaches, show the equivalence between the Hurwitz stability of matrix A^+, the existence of several classes of Lyapunov functions, and the existence of several classes of exponentially decreasing sets that are positively invariant with respect to the interval system dynamics.**

Keywords—Interval systems, positive systems, common Lyapunov functions, positively invariant sets

I. INTRODUCTION

A. Research context

The concept of row and column representatives associated with a nonempty set of matrices was introduced in [1] and [2] that exclusively focused on the algebraic properties corresponding to P (P0) matrices. The connection to dynamical problems was made at the end of the decade 2000–2010 by paper [3] interested in Lyapunov functions for switched positive linear systems. During the same period, column representatives have been used by several noticeable works studying the characterization of linear copositive Lyapunov functions for arbitrary switching positive linear systems with continuous-time dynamics [4], [5], or discrete-time dynamics [6]. For both cases, paper [7] presented results on max-type Lyapunov functions that relied on row representatives.

Our recent paper [8] constructed a global picture of the use of row and column representatives in the qualitative analysis of arbitrary switching positive systems. Stimulated by the mentioned research, in this paper, we are going to explore the dynamics of positive interval systems, by applying the theory of representatives to the finite family of interval matrix vertices.

Towards this goal, we consider the *positive interval system*

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t),\ \boldsymbol{x}(t_0)=\boldsymbol{x}_0,\ t,t_0\in\mathbb{R}_+,\ t\ge t_0,\ \boldsymbol{A}\in\mathbb{A}, \quad (1)$$

where

$$\mathbb{A}=\left\{\boldsymbol{A}\in\mathbb{R}^{n\times n}\mid \boldsymbol{A}^-\le\boldsymbol{A}\le\boldsymbol{A}^+\right\}\overset{not}{=}[\boldsymbol{A}^-,\ \boldsymbol{A}^+] \quad (2)$$

is an interval matrix defined by componentwise inequalities (see notations in Section II), and each element $\boldsymbol{A}\in\mathbb{A}$ is a *Metzler* matrix, or, equivalently, an *essentially nonnegative* matrix (with all off-diagonal entries nonnegative). The key particularity in the dynamics of system (1)&(2) is the behavior of any trajectory initiated in the *positive orthant*, which remains therein forever (e. g. [9], [10]).

On the other hand, the dynamics of interval systems of general form (not necessarily positive) was addressed by exploiting the properties of a dominant matrix that majorizes all matrices belonging to the interval matrix. A series of important ideas related to the diagonal stability of interval systems can be found in the monograph [11], chapter 3 and the references therein. Significant expansions of these ideas were later reported by our articles [12] (that refers strictly to interval matrices) and [13] (that refers to matrix polytopes, where interval matrices represent a particular case). Within the context of polytopic systems, it is worth mentioning the results presented by [4] for the existence of copositive Lyapunov functions associated with switching and polytopic systems, respectively.

B. Paper objectives and organization

Our current work intends to develop a comparative study for the exploration of positive interval systems dynamics, by the two types of techniques briefly summarized above:

• techniques relying on the properties of row and column representatives associated with the set of the interval matrix vertices:

$$\mathcal{A}=\left\{\boldsymbol{A}^1,\ldots,\boldsymbol{A}^K\right\}\subset\mathbb{R}^{n\times n},\ K\le 2^{n^2}\ ; \quad (3)$$

• techniques relying on the properties of the dominant vertex $\boldsymbol{A}^+$ of the interval matrix, which ensures the fulfillment of the componentwise inequalities:

$$\forall \boldsymbol{A}^\theta\in\mathcal{A}:\boldsymbol{A}^\theta\le\boldsymbol{A}^+,\theta=1,\ldots,K\ . \quad (4)$$

Our exposition is organized as follows. The notations used for presentation are detailed in Section II. Section III is devoted to analysis tools derived from the theory of row and column representatives associated with the vertex set (3). Section IV is dedicated to analysis tools derived from matrix measure inequalities associated with the matrix inequalities (4). Section V constructs a comparative analysis of the techniques exploited by Sections III and IV, and emphasizes the role of the properties of the dominant vertex $\boldsymbol{A}^+$ – unlike arbitrary polytopic systems that, in general, do not have a vertex exhibiting such properties. Section V formulates some concluding remarks on the research progress supported by the current work.

II. PREREQUISITES

We use the notation $\| \ \|_p$ for both the Hölder vector p-norm, as well as the induced matrix p-norm. The value $\mu_p(\boldsymbol{M}) = \lim_{h \downarrow 0} \frac{1}{h}(\| \boldsymbol{I} + h\boldsymbol{M} \|_p - 1)$ is the matrix measure based on the matrix norm $\| \ \|_p$ - see ([14], Fact 4.11.5).

For vectors as well as for matrices, the notation $\bullet^T$ means transposition. A matrix $\boldsymbol{M} \in \mathbb{R}^{n \times q}$ is called: • *nonnegative*, $\boldsymbol{M} \geq 0$, if all its entries are nonnegative; • *semi-positive*, $\boldsymbol{M} > 0$, if it is nonnegative and at least an entry is positive; • *positive*, $\boldsymbol{M} \gg 0$, if all its entries are positive. For $\boldsymbol{M}_1, \boldsymbol{M}_2 \in \mathbb{R}^{n \times q}$, the matrix inequalities $\boldsymbol{M}_1 \geq \boldsymbol{M}_2$, $\boldsymbol{M}_1 > \boldsymbol{M}_2$ and $\boldsymbol{M}_1 \gg \boldsymbol{M}_2$ mean $\boldsymbol{M}_1 - \boldsymbol{M}_2 \geq 0$, $\boldsymbol{M}_1 - \boldsymbol{M}_2 > 0$ and, respectively, $\boldsymbol{M}_1 - \boldsymbol{M}_2 \gg 0$. A square matrix $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is called: • *essentially-nonnegative* (*Metzler*) if all its off-diagonal entries are nonnegative; • *essentially-positive* if all its off-diagonal entries are positive. If matrix $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is symmetric, then $\boldsymbol{M} \succ 0$ ($\boldsymbol{M} \prec 0$) means "$\boldsymbol{M}$ is positive (negative) definite".

For a square matrix $\boldsymbol{M} \in \mathbb{R}^{n \times n}$, let us denote its eigenvalues by $\lambda_i(\boldsymbol{M}) \in \mathbb{C}$, $i = 1, \ldots, n$. If $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is (*essentially*) *nonnegative*, then: • $\boldsymbol{M}$ has a real eigenvalue $\lambda_{\max}(\boldsymbol{M})$ so that: (i) $|\lambda_i(\boldsymbol{M})| \leq \lambda_{\max}(\boldsymbol{M})$, $i = 1, \ldots, n$ (for $\boldsymbol{M}$ nonnegative); (ii) $\mathrm{Re}\{\lambda_i(\boldsymbol{M})\} \leq \lambda_{\max}(\boldsymbol{M})$, $i = 1, \ldots, n$ (for $\boldsymbol{M}$ essentially-nonnegative). • $\boldsymbol{M}$ has semipositive right and left eigenvectors $\boldsymbol{r}(\boldsymbol{M}) > 0$, $\boldsymbol{\ell}(\boldsymbol{M}) > 0$, corresponding to $\lambda_{\max}(\boldsymbol{M})$. • If $\boldsymbol{M}$ is *irreducible* (the associated graph is strongly connected) then $\lambda_{\max}(\boldsymbol{M})$ is a simple eigenvalue and the associated eigenvectors are positive $\boldsymbol{r}(\boldsymbol{M}) \gg 0$, $\boldsymbol{\ell}(\boldsymbol{M}) \gg 0$ (see ([14], Fact 4.11.5).

Let $\mathcal{M} = \{\boldsymbol{M}^1, \ldots, \boldsymbol{M}^K\} \subset \mathbb{R}^{n \times n}$ be a set of matrices. For $\pi : \{1, \ldots, n\} \to \{1, \ldots, K\}$, let $\boldsymbol{\pi} = (\pi(1), \ldots, \pi(n))$; denote by $\boldsymbol{\Pi}$ the set of all n-tuples with values in $\{1, \ldots, K\}$. For $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, the matrix $\bar{\boldsymbol{M}}_{\boldsymbol{\pi}} = \begin{bmatrix} \mathrm{row}_1(\boldsymbol{M}^{\pi(1)}) \\ \ldots \\ \mathrm{row}_n(\boldsymbol{M}^{\pi(n)}) \end{bmatrix} \in \mathbb{R}^{n \times n}$ is a *row representative* of the matrix set $\mathcal{M}$; the set of all row representatives is denoted by $\overline{\mathcal{M}}^{\#}$. Similarly, the matrix $\underline{\boldsymbol{M}}_{\boldsymbol{\pi}} = \left[\mathrm{column}_1(\boldsymbol{M}^{\pi(1)}) \ \ldots \ \mathrm{column}_n(\boldsymbol{M}^{\pi(n)})\right] \in \mathbb{R}^{n \times n}$ is a *column representative* of the matrix set $\mathcal{M}$; the set of all column representatives is denoted by $\underline{\mathcal{M}}^{\#}$. If all matrices from $\mathcal{M}$ are (essentially) nonnegative, then all matrices from $\overline{\mathcal{M}}^{\#}$ and $\underline{\mathcal{M}}^{\#}$ are also (essentially) nonnegative [3].

III. RESULTS DERIVED FROM VERTEX REPRESENTATIVES

This section presents analysis instruments for the dynamics of positive interval systems derived from the properties of the row / column representatives associated with the vertex set $\mathcal{A}$ (3) of the interval matrix $\mathbb{A}$ (2). The proposed results rely on the following lemma, which refers to an arbitrary set of essentially nonnegative matrices (not necessarily the vertices of an interval matrix).

Lemma 1.

Let

$$\mathcal{M} = \{\boldsymbol{M}^1, \ldots, \boldsymbol{M}^K\} \subset \mathbb{R}^{n \times n} \tag{5}$$

be a set of *essentially nonnegative matrices*.

(a) Consider the greatest eigenvalue of all row representatives $\overline{\mathcal{M}}^{\#}$ associated with $\mathcal{M}$ (5)

$$\bar{\lambda}^* = \max_{\boldsymbol{\pi} \in \boldsymbol{\Pi}} \lambda_{\max}(\bar{\boldsymbol{M}}_{\boldsymbol{\pi}}) \tag{6}$$

which corresponds to one or several row representatives from the set $\overline{\mathcal{M}}^* = \left\{\bar{\boldsymbol{M}}^*_{\boldsymbol{\pi}} \in \overline{\mathcal{M}}^{\#} \mid \lambda_{\max}(\bar{\boldsymbol{M}}^*_{\boldsymbol{\pi}}) = \bar{\lambda}^*\right\}$. For any $\varepsilon > 0$, there exist $\bar{s} \in [\bar{\lambda}^*, \bar{\lambda}^* + \varepsilon)$ and a positive vector $\bar{\boldsymbol{w}} \in \mathbb{R}^n$, $\bar{\boldsymbol{w}} \gg 0$, such that

$$\boldsymbol{M}^{\theta} \bar{\boldsymbol{w}} \leq \bar{s} \, \bar{\boldsymbol{w}}, \theta = 1, \ldots, K \tag{7}$$

(b) Consider the greatest eigenvalue of the column representatives $\underline{\mathcal{M}}^{\#}$ associated with $\mathcal{M}$ (5)

$$\underline{\lambda}^* = \max_{\boldsymbol{\pi} \in \boldsymbol{\Pi}} \lambda_{\max}(\underline{\boldsymbol{M}}_{\boldsymbol{\pi}}) \tag{8}$$

which corresponds to one or several representatives from the set $\underline{\mathcal{M}}^* = \left\{\underline{\boldsymbol{M}}^*_{\boldsymbol{\pi}} \in \underline{\mathcal{M}}^{\#} \mid \lambda_{\max}(\underline{\boldsymbol{M}}^*_{\boldsymbol{\pi}}) = \underline{\lambda}^*\right\}$.

For any $\varepsilon > 0$, there exist $\underline{s} \in [\underline{\lambda}^*, \underline{\lambda}^* + \varepsilon)$ and a positive vector $\underline{\boldsymbol{w}} \in \mathbb{R}^n$, $\underline{\boldsymbol{w}} \gg 0$, such that

$$\underline{\boldsymbol{w}}^T \boldsymbol{M}^{\theta} \leq \underline{s} \, \underline{\boldsymbol{w}}^T, \ \theta = 1, \ldots, K \tag{9}$$

Proof: (a) If the set $\overline{\mathcal{M}}^*$ contains an irreducible matrix $\bar{\boldsymbol{M}}^*_{\boldsymbol{\pi}}$, then the pair $(\bar{s}, \bar{\boldsymbol{w}})$, with $\bar{s} = \bar{\lambda}^* = \lambda_{\max}(\bar{\boldsymbol{M}}^*_{\boldsymbol{\pi}})$ and $\bar{\boldsymbol{w}} = \boldsymbol{r}(\bar{\boldsymbol{M}}^*_{\boldsymbol{\pi}})$ represent the unique solution to inequalities (8), as per ([7], Lemma 1). If all the matrices in $\overline{\mathcal{M}}^*$ are reducible, then $(\bar{s}, \bar{\boldsymbol{w}})$ can be defined by using the row representatives associated with the set $\boldsymbol{M}^{\theta}(c) = \boldsymbol{M}^{\theta} + c\,\boldsymbol{E}$, $\theta = 1, \ldots, K$, where $c > 0$, and $\boldsymbol{E} = \left[e_{ij}\right] \in \mathbb{R}^{n \times n}$ with $e_{ii} = 0$ and $e_{ij} = 1$ for $i \neq j$. The set $\mathcal{M}(c)$ means a slightly perturbed form of $\mathcal{M}$ (5), where all matrices are essentially positive (instead of essentially nonnegative) and, consequently, irreducible. Thus, any matrix $\overline{\boldsymbol{M}(c)}^*_{\boldsymbol{\pi}} \in \overline{\mathcal{M}(c)}^*$ can provide $\bar{s} = \overline{\lambda(c)}^* = \lambda_{\max}(\overline{\boldsymbol{M}(c)}^*_{\boldsymbol{\pi}})$, $\bar{\boldsymbol{w}} = \boldsymbol{r}(\overline{\boldsymbol{M}(c)}^*_{\boldsymbol{\pi}})$ that represent the unique solution to inequalities (8) written for matrices $\boldsymbol{M}^{\theta}(c)$, i.e. $\boldsymbol{M}^{\theta}(c)\bar{\boldsymbol{w}} \leq \bar{s}\,\bar{\boldsymbol{w}}$, $\theta = 1, \ldots, K$. Subsequently, inequalities (8) written for $\boldsymbol{M}^{\theta}$ are also solved, since $\boldsymbol{M}^{\theta}\bar{\boldsymbol{w}} \leq \boldsymbol{M}^{\theta}(c)\,\bar{\boldsymbol{w}}$. Obviously, for $\varepsilon > 0$ arbitrary small, there exists $c(\varepsilon) > 0$ so as $\bar{s} = \overline{\lambda(c(\varepsilon))}^* = \lambda_{\max}(\overline{\boldsymbol{M}(c(\varepsilon))}^*_{\boldsymbol{\pi}}) < \bar{\lambda}^* + \varepsilon$.

(b) The proof is similar to the proof for (a). ■

Now we can focus on the dynamics analysis for the positive interval system (1)&(2). The use of Lemma 1 for the vertex set $\mathcal{A}$ (3) of the interval matrix $\mathbb{A}$ (2) means the following particularizations. All row (column) representatives associated with $\mathcal{A}$ (3) are essentially nonnegative matrices that satisfy the componentwise inequalities $\overline{\boldsymbol{A}}_{\boldsymbol{\pi}} \le \boldsymbol{A}^+$ and $\underline{\boldsymbol{A}}_{\boldsymbol{\pi}} \le \boldsymbol{A}^+$, $\forall \boldsymbol{\pi} \in \boldsymbol{\Pi}$, which imply $\boldsymbol{A}^+ \in \overline{\mathcal{A}}^*$, $\overline{\lambda}^* = \lambda_{\max}(\boldsymbol{A}^+)$, and $\boldsymbol{A}^+ \in \underline{\mathcal{A}}^*$, $\underline{\lambda}^* = \lambda_{\max}(\boldsymbol{A}^+)$, respectively, as per ([14], Fact 4.11.18).

Theorem 1.

Consider the positive interval system (1)&(2). The following statements are equivalent:

(i) Matrix $\boldsymbol{A}^+$ is Hurwitz stable.

(ii) All matrices $\boldsymbol{A} \in \mathbb{A}$ are Hurwitz stable.

(iii) There exist a positive vector $\overline{\boldsymbol{w}} \in \mathbb{R}^n$, $\overline{\boldsymbol{w}} \gg 0$ and a negative constant $\overline{s} < 0$ that fulfill the inequality

$$\boldsymbol{A}^+ \overline{\boldsymbol{w}} \le \overline{s}\, \overline{\boldsymbol{w}} . \tag{10}$$

(iv) There exist a positive vector $\underline{\boldsymbol{w}} \in \mathbb{R}^n$, $\underline{\boldsymbol{w}} \gg 0$, and a negative constant $\underline{s} < 0$ that fulfill the inequality

$$\underline{\boldsymbol{w}}^T \boldsymbol{A}^+ \le \underline{s}\, \underline{\boldsymbol{w}}^T . \tag{11}$$

(v) There exist a positive vector $\overline{\boldsymbol{w}} \in \mathbb{R}^n$, $\overline{\boldsymbol{w}} = [\overline{w}_1 \dots \overline{w}_n] \gg 0$, and a negative constant $\overline{s} < 0$, for which the max-type function of the form

$$\overline{\mathcal{W}} : \mathbb{R}^n_+ \to \mathbb{R}_+,\ \overline{\mathcal{W}}(\boldsymbol{x}) = \max_{i=1,n} \left\{ \frac{x_i}{\overline{w}_i} \right\}, \tag{12}$$

is a Lyapunov function for the interval system (1)&(2), satisfying the inequality

$$\begin{aligned} &\mathrm{D}_t^+ \overline{\mathcal{W}}(\boldsymbol{x}(t)) = \\ &= \lim_{h \downarrow 0} \tfrac{1}{h} \left[\overline{\mathcal{W}}(\boldsymbol{x}(t+h)) - \overline{\mathcal{W}}(\boldsymbol{x}(t)) \right] \le \overline{s}\, \overline{\mathcal{W}}(\boldsymbol{x}(t)), \end{aligned} \tag{13}$$

along any non-trivial trajectory.

(vi) There exist a positive vector $\underline{\boldsymbol{w}} \in \mathbb{R}^n$, $\underline{\boldsymbol{w}} = [\underline{w}_1 \dots \underline{w}_n] \gg 0$, and a negative constant $\underline{s} < 0$, for which the linear function of the form

$$\underline{\mathcal{W}} : \mathbb{R}^n_+ \to \mathbb{R}_+,\ \underline{\mathcal{W}}(\boldsymbol{x}) = \sum_{i=1}^n \underline{w}_i x_i = \underline{\boldsymbol{w}}^T \boldsymbol{x} , \tag{14}$$

is a Lyapunov function for the interval system (1)&(2), satisfying the inequality

$$\mathrm{D}_t^+ \underline{\mathcal{W}}(\boldsymbol{x}(t)) = \lim_{h \downarrow 0} \tfrac{1}{h} [\underline{\mathcal{W}}(\boldsymbol{x}(t+h)) - \underline{\mathcal{W}}(\boldsymbol{x}(t))] \le \underline{s}\, \underline{\mathcal{W}}(\boldsymbol{x}(t)) , \tag{15}$$

along any non-trivial trajectory.

(vii) There exist a positive vector $\overline{\boldsymbol{w}} \in \mathbb{R}^n$, $\overline{\boldsymbol{w}} = [\overline{w}_1 \dots \overline{w}_n] \gg 0$, and a negative constant $\overline{s} < 0$, for which the sets

$$\overline{\mathcal{X}}(t) = \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \,\middle|\, \max_{i=1,\dots,n} \left\{ \frac{x_i}{\overline{w}_i} \right\} \le \alpha e^{\overline{s} t} \right\},\ t \in \mathbb{R}_+,\ \alpha > 0 , \tag{16}$$

are positively invariant with respect to the trajectories of the interval system (1)&(2), i.e. any trajectory $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ of system (1) initiated at $t_0 \in \mathbb{R}_+$ in $\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \overline{\mathcal{X}}(t_0)$ satisfies $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0) \in \overline{\mathcal{X}}(t)$ for any $t \ge t_0$.

($viii$) There exist a positive vector $\underline{\boldsymbol{w}} \in \mathbb{R}^n$, $\underline{\boldsymbol{w}} = [\underline{w}_1 \dots \underline{w}_n] \gg 0$, and a negative constant $\underline{s} < 0$, for which the sets of the form

$$\underline{\mathcal{X}}(t) = \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \underline{w}_i x_i \le \alpha e^{\underline{s} t} \right\},\ t \in \mathbb{R}_+,\ \alpha > 0 , \tag{17}$$

are positively invariant with respect to the trajectories of the interval system (1)&(2).

Proof. $(i) \Rightarrow (ii)$: Consider $\sigma > 0$ so that $\sigma \boldsymbol{I} + \boldsymbol{A}^- \ge 0$. Hence $\sigma \boldsymbol{I} + \boldsymbol{A}^+ \ge \sigma \boldsymbol{I} + \boldsymbol{A} \ge 0$, $\forall \boldsymbol{A} \in \mathbb{A}$, that implies $\sigma + \lambda_{\max}(\boldsymbol{A}) \le \sigma + \lambda_{\max}(\boldsymbol{A}^+)$, according to ([14], Fact 4.11.18), meaning that $\lambda_{\max}(\boldsymbol{A}) \le \lambda_{\max}(\boldsymbol{A}^+) < 0$, $\forall \boldsymbol{A} \in \mathbb{A}$.

$(ii) \Rightarrow (i)$: It is obvious, since $\boldsymbol{A}^+ \in \mathbb{A}$.

$(i) \Rightarrow (iii)$: Applying Lemma 1 with $\overline{\lambda}^* = \lambda_{\max}(\boldsymbol{A}^+) < 0$, there exists $\varepsilon > 0$ so that inequalities $\boldsymbol{A}^{\theta} \overline{\boldsymbol{w}} \le \overline{s}\, \overline{\boldsymbol{w}}$, $\theta = 1,\dots,K$, are satisfied with $\overline{\lambda}^* \le \overline{s} < \overline{\lambda}^* + \varepsilon < 0$, and $\overline{\boldsymbol{w}} \in \mathbb{R}^n$, $\overline{\boldsymbol{w}} \gg 0$. This pair $(\overline{s}, \overline{\boldsymbol{w}})$ also solves (10), which represents a subset of inequalities $\boldsymbol{A}^{\theta} \overline{\boldsymbol{w}} \le \overline{s}\, \overline{\boldsymbol{w}}$, $\theta = 1,\dots,K$.

$(iii) \Rightarrow (i)$: Consider $\sigma > 0$ so that $\sigma \boldsymbol{I} + \boldsymbol{A}^+ \ge 0$. Inequality (10) implies $(\sigma \boldsymbol{I} + \boldsymbol{A}^+) \overline{\boldsymbol{w}} \le (\sigma + \overline{s}) \overline{\boldsymbol{w}}$ which ensures $\sigma + \lambda_{\max}(\boldsymbol{A}^+) = \lambda_{\max}(\sigma I + \boldsymbol{A}^+) \le \sigma + \overline{s}$ as per ([15], Corollary 8.1.29), meaning $\lambda_{\max}(\boldsymbol{A}^+) \le \overline{s} < 0$.

$(i) \Rightarrow (v)$: Applying Lemma 1 with $\overline{\lambda}^* = \lambda_{\max}(\boldsymbol{A}^+) < 0$, there exists $\varepsilon > 0$ so that inequalities $\boldsymbol{A}^{\theta} \overline{\boldsymbol{w}} \le \overline{s}\, \overline{\boldsymbol{w}}$, $\theta = 1,\dots,K$, are satisfied with $\overline{\lambda}^* \le \overline{s} < \overline{\lambda}^* + \varepsilon < 0$, and $\overline{\boldsymbol{w}} \in \mathbb{R}^n$, $\overline{\boldsymbol{w}} \gg 0$. In other words, the inequality $\boldsymbol{A} \overline{\boldsymbol{w}} \le \overline{s}\, \overline{\boldsymbol{w}}$ is fulfilled for any vertex of the interval matrix $\mathbb{A}$ (2), and we are going to show that the inequality holds true for any matrix $\boldsymbol{A} \in \mathbb{A}$ due to the convexity of $\mathbb{A}$. For any $\boldsymbol{A} \in \mathbb{A}, \exists \gamma^{\theta} \ge 0$, $\sum_{\theta=1}^K \gamma^{\theta} = 1$, such that $\boldsymbol{A} = \sum_{\theta=1}^K \gamma^{\theta} \boldsymbol{A}^{\theta}$, therefore

$$\boldsymbol{A} \overline{\boldsymbol{w}} = \left(\sum_{k=1}^K \gamma^{\theta} \boldsymbol{A}^{\theta} \right) \overline{\boldsymbol{w}} = \sum_{k=1}^K \gamma^{\theta} (\boldsymbol{A}^{\theta} \overline{\boldsymbol{w}}) \le \sum_{k=1}^K \gamma^{\theta} (\overline{s}\, \overline{\boldsymbol{w}}) = \overline{s}\, \overline{\boldsymbol{w}} .$$

On the other hand, given a trajectory of system (1)&(2) corresponding to an arbitrary $\boldsymbol{A} \in \mathbb{A}$, for the function $\overline{\mathcal{W}}(\boldsymbol{x}(t))$, at any $\tau \in \mathbb{R}_+$ there exists $\overline{\tau} > \tau$ and $k \in \{1,\dots,n\}$ so that $\forall t \in [\tau, \overline{\tau}) : \overline{\mathcal{W}}(\boldsymbol{x}(t)) = x_k(t) / \overline{w}_k$. Subsequently, for $t \in [\tau, \overline{\tau})$ the following inequality holds true

$$\begin{aligned} \mathrm{D}_t^+ \overline{\mathcal{W}}(\boldsymbol{x}(t)) &= \frac{\dot{x}_k(t)}{\overline{w}_k} = \frac{1}{\overline{w}_k} (\boldsymbol{A}\boldsymbol{x})_k = \frac{1}{\overline{w}_k} \sum_{j=1}^n a_{kj} \overline{w}_j \frac{x_j(t)}{\overline{w}_j} \\ &\le \frac{1}{\overline{w}_k} \left[\sum_{j=1}^n a_{kj} \overline{w}_j \right] \max_{j=1,n} \left\{ \frac{x_j(t)}{\overline{w}_j} \right\} \le \left[\frac{1}{\overline{w}_k} \overline{s} \overline{w}_k \right] \overline{\mathcal{W}}(\boldsymbol{x}(t)) \le \overline{s}\, \overline{\mathcal{W}}(\boldsymbol{x}(t)). \end{aligned}$$

$(v) \Rightarrow (i)$: Along any trajectory generated by $A^+ \in \mathbb{A}$, the function $\mathcal{W}(x(t))$ (12) satisfies (13), showing that the equilibrium $\{0\}$ of system (1)&(2) with $A = A^+$ is exponentially stable. Hence, matrix A^+ is Hurwitz stable.
$(v) \Rightarrow (vii)$: Inequality (13) implies that for any trajectory $x(t) = x(t;0,x^0)$ initiated in arbitrary $x^0 = x(0) \geq 0$, the function $\mathcal{W}(x(t))$ satisfies $\mathcal{W}(x(t)) \leq e^{st}\mathcal{W}(x_0)$, according to ([16], Theorem 4.2.11). Hence, for any $A \in \mathbb{A}$ and for any $x^0 \geq 0$ satisfying $\max\{x_i^0 / \bar{w}_i\} = \mathcal{W}(x^0) \leq \alpha$, we get $\mathcal{W}(x(t)) \leq e^{st}\mathcal{W}(x^0) \leq \alpha\, e^{st}$, for all $t \geq 0$. Thus, any trajectory initiated inside the set (16), remains inside this set.
$(vii) \Rightarrow (v)$: If the set (16) is invariant, consider a trajectory of system (1)&(2) corresponding to some $A \in \mathbb{A}$, initiated in $x^0 = x(0) \geq 0$, so that $\mathcal{W}(x^0) = \alpha$. Along this trajectory, the inequality $\mathcal{W}(x(t)) \leq \alpha e^{st} = \mathcal{W}(x^0)\, e^{st}$ holds true, which implies (13), in accordance with ([16], Theorem 4.2.11). The proof is completed, since the trajectory is arbitrary.
Similarly, $(i) \Leftrightarrow (iv)$, $(i) \Leftrightarrow (vi)$ and $(vi) \Leftrightarrow (viii)$. ∎

IV. Results Derived From Vertex Majorizations

This section presents analysis instruments for the dynamics of positive interval systems derived from the majorization (4) of the vertex matrices belonging to the set $\mathcal{A}$ (3) of the interval matrix $\mathbb{A}$ defined by (2). The proposed results rely on the following lemma, which refers to two arbitrary essentially nonnegative matrices (not necessarily vertices of an interval matrix).

Lemma 2.

Let $1 \leq p \leq \infty$, $v = [v_1 \dots v_n]^T \gg 0$ and denote by $V = \mathrm{diag}\{v\}$ the diagonal matrix with the entries defined by the positive vector v. Consider two matrices $M, P \in \mathbb{R}^{n\times n}$ that are essentially nonnegative and satisfy the componentwise inequality $M \leq P$. Then

$$\mu_p(V^{-1}MV) \leq \mu_p(V^{-1}PV) \ . \qquad (18)$$

Proof: Considering a small $h > 0$, we get
$0 \leq V^{-1}(I + hM)V \leq V^{-1}(I + hP)V$,
which, for any $y \in \mathbb{R}^n$, leads to the vector inequality
$|(V^{-1}(I + hM)V)|\, y\,|\,| \leq |(V^{-1}(I + hP)V)|\, y\,|\,|$.
Relying on the monotonicity of the vector p-norm ([15], Theorem 5.5.10) we get
$\|(V^{-1}(I + hM)V)y\|_p \leq \|(V^{-1}(I + hM)V)\,|\,y\,|\,\|_p \leq$
$\|(V^{-1}(I + hP)V)\,|\,y\,|\,\|_p \leq \|V^{-1}(I + hP)V\|\,\|y\|_p$.
Taking $\|y\|_p = 1$, we get the matrix norm inequality
$\|V^{-1}(I + hM)V\|_p = \max_{\|y\|_p = 1} \|(V^{-1}(I + hM)V)y\|_p \leq$
$\leq \|V^{-1}(I + hP)V\|_p$,
which implies
$(\|I + hV^{-1}MV\|_p - 1)/h \leq (\|I + hV^{-1}PV\|_p - 1)/h$.
By using $h \downarrow 0$, the inequality $\mu_p(V^{-1}MV) \leq \mu_p(V^{-1}PV)$ is obtained, meaning that (18) is true. ∎

Theorem 2

Consider the positive interval system (1)&(2). The following statements are equivalent:
(i) Matrix A^+ is Hurwitz stable.
(ii) All matrices $A \in \mathbb{A}$ are Hurwitz stable.
(iii) There exists p, $1 \leq p \leq \infty$, so as there exist a positive vector $v_p \in \mathbb{R}^n$, $v_p \gg 0$, and a negative constant $s_p < 0$ so that matrix $V_p = \mathrm{diag}\{v_p\}$ fulfills the inequality

$$\mu_p(V_p^{-1}A^+V_p) \leq s_p \ . \qquad (19)$$

(iv) For any p, $1 \leq p \leq \infty$, there exist a positive vector $v_p \in \mathbb{R}^n$, $v_p \gg 0$, and a negative constant $s_p < 0$ that matrix $V_p = \mathrm{diag}\{v_p\}$ fulfills inequality (19).
(v) There exists p, $1 \leq p \leq \infty$, so that there exist a positive vector $v_p \in \mathbb{R}^n$, $v_p \gg 0$, and a negative constant $s_p < 0$, for which the function

$$\mathcal{V}_p : \mathbb{R}_+^n \to \mathbb{R}_+,\ \mathcal{V}_p(x) = \|V_p^{-1}x\|_p \ , \qquad (20)$$

with $V_p = \mathrm{diag}\{v_p\}$, is a Lyapunov function for the interval system (1)&(2), satisfying the inequality

$$\begin{aligned} &\mathrm{D}_t^+\mathcal{V}_p(x(t)) = \\ &= \lim_{h\downarrow 0} \tfrac{1}{h}\left[\mathcal{V}_p(x(t+h)) - \mathcal{V}_p(x(t))\right] \leq s_p\, \mathcal{V}_p(x(t)) \end{aligned} \qquad (21)$$

along any non-trivial trajectory.
(vi) For any p, $1 \leq p \leq \infty$, there exist a positive vector $v_p \in \mathbb{R}^n$, $v_p \gg 0$, and a negative constant $s_p < 0$ for which $\mathcal{V}_p$ (20) is a Lyapunov function for the interval system (1)&(2), satisfying inequality (21) along any non-trivial trajectory.
(vii) There exists p, $1 \leq p \leq \infty$, so as there exist a positive vector $v_p \in \mathbb{R}^n$, $v_p \gg 0$, and a negative constant $s_p < 0$, for which the set

$$\mathcal{X}_p(t) = \left\{x \in \mathbb{R}_+^n \,\middle|\, \|V_p^{-1}x\|_p \leq \alpha e^{s_p t}\right\},\ t \in \mathbb{R}_+, \alpha > 0, \qquad (22)$$

is positively invariant with respect to the trajectories of the interval system (1)&(2).
($viii$) For any p, $1 \leq p \leq \infty$, there exist a positive vector $v_p \in \mathbb{R}^n$, $v_p \gg 0$, and a negative constant $s_p < 0$ for which the set $\mathcal{X}_p(t)$ (22) is invariant with respect to the trajectories of the interval system (1)&(2).
Proof. $(i) \Leftrightarrow (ii)$: It is similar to the proof of Theorem 1.
$(i) \Rightarrow (iv)$: We first prove that for any $\varepsilon > 0$, there exist

$s_p \in [\lambda_{\max}(A^+), \lambda_{\max}(A^+)+\varepsilon)$ and $v_p \in \mathbb{R}^n$, $v_p \gg 0$, such that $\mu_p(V_p^{-1}A^+V_p) \le s_p$, with $V_p = \mathrm{diag}\{v_p\}$. Consider $\sigma > 0$ satisfying $\sigma I + A^+ \ge 0$. Assuming that A^+ is *irreducible*, the left, and right Perron-Frobenius eigenvectors of A^+ are positive, $\ell(A^+) = [\ell_1 \dots \ell_n]^T \gg 0$, $r(A^+) = [r_1 \dots r_n]^T \gg 0$. The theorem presented by [17] yields $\| V_p^{-1}(A^+ + \sigma I)V_p \|_p = \lambda_{\max}(A^+ + \sigma I)$, with $v_p = [r_1^{1/q} / \ell_1^{1/p} \cdots r_n^{1/q} / \ell_n^{1/p}]^T \gg 0$ where $1/p + 1/q = 1$, $1 \le q \le \infty$ (with $1/p = 1, 1/q = 0$, and $1/p = 0, 1/q = 1$). Thus, for the left hand side of (19) we write

$$\mu_p(V_p^{-1}A^+V_p) = \lim_{h \downarrow 0}(\| V_p^{-1}(I + hA^+)V_p \|_p - 1)/h =$$
$$= \lim_{\sigma \to \infty}(\lambda_{\max}(V_p^{-1}(A^+ + \sigma I)V_p) - \sigma) = \lambda_{\max}(V_p^{-1}A^+V_p)$$

Since $\lambda_{\max}(V_p^{-1}A^+V_p) = \lambda_{\max}(A^+)$, inequality (19) is satisfied as equality, with $s_p = \lambda_{\max}(A^+) < 0$ and $v_p \gg 0$.

If matrix A^+ is *reducible*, then the above approach can be used for the irreducible matrix $A^+(c) = A^+ + cE$, where $c > 0$, and $E = \left[e_{ij}\right] \in \mathbb{R}^{n \times n}$ with $e_{ii} = 0$ and $e_{ij} = 1$ for $i \ne j$. For $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ so as $s_p = \lambda_{\max}(A^+(c(\varepsilon)) < \lambda_{\max}(A^+) + \varepsilon$, in the sense that s_p can be taken as close to $\lambda_{\max}(A^+)$ as we want. By considering $v_p(c) = [r_1^{1/q} / \ell_1^{1/p} \cdots r_n^{1/q} / \ell_n^{1/p}]^T \gg 0$ defined by the entries of the positive eigenvectors of $A^+(c)$, i.e. $\ell(A^+(c)) = [\ell_1 \dots \ell_n]^T \gg 0$, $r(A^+(c)) = [r_1 \dots r_n]^T \gg 0$, we can write $\mu_p((V_p(c))^{-1}A^+(c)V_p(c)) = s_p$, where $V_p(c) = \mathrm{diag}\{v_p(c)\}$. On the other hand, Lemma 2 ensures $\mu_p((V(c))^{-1}A^+V_p(c)) \le \mu_p((V(c))^{-1}A^+(c)V_p(c))$. Since A^+ is Hurwitz, i.e. $\lambda_{\max}(A^+) < 0$, there exists $s_p < 0$, such that (19) holds true with v_p or $v_p(c)$ discussed above.

$(iv) \Rightarrow (iii)$: It is obvious.

$(iii) \Rightarrow (i)$: It results from the inequality $\lambda_{\max}(A^+) = \lambda_{\max}(V_p^{-1}A^+V_p) \le \mu_p(V_p^{-1}A^+V_p) < 0$.

$(iv) \Rightarrow (vi)$: Consider an arbitrary p, $1 \le p \le \infty$, and $\mathcal{V}_p(x)$ of form (20), where the diagonal, positive definite matrix V_p satisfies (19). Along any nontrivial trajectory of system (1)&(2) (corresponding to an arbitrary $A \in \mathbb{A}$) the function $\mathcal{V}_p(x)$ is positive definite, and for the Dini derivative we can write $D^+\mathcal{V}_p(x(t)) = \lim_{h \downarrow 0} \frac{1}{h}\left[\mathcal{V}_p(x(t+h)) - \mathcal{V}_p(x(t))\right] =$

$$= \lim_{h \downarrow 0} \tfrac{1}{h}(\| V_p^{-1}e^{Ah}x(t) \|_p - \| V_p^{-1}x(t) \|_p) \le$$
$$\le \lim_{h \downarrow 0} \tfrac{1}{h}(\| V_p^{-1}e^{Ah}V_p \|_p \| V_p^{-1}x(t) \|_p - \| V_p^{-1}x(t) \|_p) =$$
$$= \left[\lim_{h \downarrow 0} \tfrac{1}{h}(\| V_p^{-1}e^{Ah}V_p \|_p - 1)\right] \| V_p^{-1}x(t) \|_p =$$
$$= \mu_p(V_p^{-1}AV_p) \| V_p^{-1}x(t) \|_p \le$$
$$\le \mu_p(V_p^{-1}A^+V_p) \| V_p^{-1}x(t) \|_p \le s_p \mathcal{V}_p(x(t))$$

Subsequently, inequality (21) is satisfied.

$(vi) \Rightarrow (v)$: It is obvious.

$(v) \Rightarrow (iii)$: Consider an arbitrary non-trivial trajectory $x(t)$ generated by A^+, which is initiated in $x^0 \ge 0$. Inequality (21) implies $\| V_p^{-1}x(t) \|_p \le e^{s_p t} \| V_p^{-1}x^0 \|_p$, for any $t \in \mathbb{R}_+^n$ and $x^0 \ge 0$, as per ([16], Theorem 4.2.11). Hence,

$$\| e^{hV_p^{-1}A^+V_p} \|_p = \| V_p^{-1}e^{hA^+}V_p \|_p =$$
$$= \sup_{x^0 \ne 0} \frac{\| (V_p^{-1}e^{hA^+}V_p)(V_p^{-1}x^0) \|_p}{\| V_p^{-1}x^0 \|_p} = \sup_{x^0 \ne 0} \frac{\| V_p^{-1}x(h) \|_p}{\| V_p^{-1}x^0 \|_p} \le e^{s_p h}$$

which yields

$$\mu_p(V_p^{-1}A^+V_p) = \lim_{h \downarrow 0} \tfrac{1}{h}\left(\| e^{hV_p^{-1}A^+V_p} \|_p - 1\right) \le$$

$\le \lim_{h \downarrow 0} \frac{1}{h}\left(e^{s_p h} - 1\right) = s_p$, showing that (19) is satisfied.

$(vi) \Rightarrow (viii)$ It is similar, *mutatis mutandis*, to the proof of implication $(v) \Rightarrow (vii)$ of Theorem 1.

$(viii) \Rightarrow (vii)$ It is obvious.

$(vii) \Rightarrow (v)$ It is similar, *mutatis mutandis*, to the proof of implication $(vii) \Rightarrow (v)$ of Theorem 1. ∎

V. Discussion on Techniques of Section III vs. Section IV

A. *Noticeable results for interval systems*

Both Theorem 1 and Theorem 2 show that a series of dynamical properties of the interval system (1)&(2) can be characterized (by equivalence) via the algebraic properties of the vertex A^+. Taking into account that (i), (ii) are identical in Theorem 1 and Theorem 2 we still have to discuss the connections clustered below in groups:

(a) The solvability of the inequalities of forms presented by Theorem 1 (iii), (iv) vs. Theorem 2 (iii), (iv) is related to the following particular cases of measure inequality (19):

• (19) for $p = \infty$ is equivalent to (10) with $v_\infty = \bar{w} \gg 0$, $s_\infty = \bar{s} < 0$; • (19) for $p = 1$ is equivalent to (11) with $v_1 = \underline{w} \gg 0$, $s_1 = \underline{s} < 0$.

(b) The existence of Lyapunov functions of forms presented by Theorem 1 (v), (vi) vs. Theorem 2 (v), (vi) is related to the following particular expressions of functions (20), satisfying inequality (21):

• (20)&(21) for $p = \infty$ is equivalent to (12)&(13) with $\mathcal{V}_\infty(x) = \bar{\mathcal{W}}(x)$, where $v_\infty = \bar{w} \gg 0$, $s_\infty = \bar{s} < 0$.

• (20)&(21) for $p = 1$ is equivalent to (14)&(15) with $\mathcal{V}_1(x) = \underline{\mathcal{W}}(x)$, where $v_1 = \underline{w} \gg 0$, $s_1 = \underline{s} < 0$.

(c) The existence of invariant sets of forms presented by Theorem 1 (vii), (viii) vs. Theorem 2 (vii), (viii) is related to the following particular shapes of exponentially decreasing sets (22):

• (22) for $p=\infty$ is equivalent to (16) with $\mathcal{X}_\infty(\boldsymbol{x})=\overline{\mathcal{X}}(\boldsymbol{x})$, where $\boldsymbol{v}_\infty=\overline{\boldsymbol{w}}\gg 0$, $s_\infty=\overline{s}<0$.

• (22) for $p=1$ is equivalent to (17) with $\mathcal{X}_1(\boldsymbol{x})=\underline{\mathcal{X}}(\boldsymbol{x})$, where $\boldsymbol{v}_1=\underline{\boldsymbol{w}}\gg 0$, $s_1=\underline{s}<0$.

The above comparative analysis shows that despite the completely different background of Theorems 1 and 2, the former may be seen as a particular case of the latter, corresponding to the $p\in\{1,\infty\}$. This is because the role of $\boldsymbol{A}^+$ in the vertex set $\mathcal{A}$ (3) of the interval matrix $\mathbb{A}$ allows one to use either tools typical to set representatives (Section III), or tools typical to matrix majorizations (Section IV).

B. Modest results for arbitrary polytopic systems

If $\mathbb{A}$ is a matrix polytope with arbitrary structure (not an interval matrix) then the overlapping of the tools developed by Sections III and IV does not hold any more. Theorem 2 (that provides generous results for interval matrices) is not valid for matrix polytopes (except for the rare structures with a dominant vertex– as analyzed by our previous work [13]). Generally speaking, the dominant eigenvalue of the row representatives $\overline{\lambda}^*$ differs from the dominant eigenvalue of the column representatives $\underline{\lambda}^*$, being provided by different matrices. Subsequently, Theorem 1 can be split into two separate parts, as follows (where the notations are preserved as in Section III). Part 1: if the dominant eigenvalue $\overline{\lambda}^*$ is given by a single row representative denoted $\boldsymbol{A}^+$, then the equivalence $(i)\Leftrightarrow(iii)\Leftrightarrow(v)\Leftrightarrow(vii)$ holds true. Part 2: if the dominant eigenvalue $\underline{\lambda}^*$ is given by a single column representative denoted $\boldsymbol{A}^+$, then $(i)\Leftrightarrow(iv)\Leftrightarrow(vi)\Leftrightarrow(viii)$ holds true.

VI. Conclusions

The current paper develops a comparative study for exploring the dynamics of a positive interval system defined by an interval matrix $\mathbb{A}=[\boldsymbol{A}^-, \boldsymbol{A}^+]\subset\mathbb{R}^{n\times n}$, by two types of techniques. The first type of techniques is based on row and column representatives corresponding to the set of interval matrix vertices; the results point out the equivalence between • the Hurwitz stability of matrix $\boldsymbol{A}^+$, • the existence of linear-type and max-type Lyapunov functions, • the existence of hyper-rhombic and -rectangular exponentially decreasing sets that are positively invariant with respect to the interval system dynamics. The second type of techniques employs the properties of the dominant vertex $\boldsymbol{A}^+$ of the positive interval matrix $\mathbb{A}$; the results show the equivalence between • the Hurwitz stability of matrix $\boldsymbol{A}^+$, • the fulfillment of a matrix measure inequality corresponding to arbitrary Hölder norms, • the existence of Lyapunov functions defined by weighted vector norms, • the existence of exponentially decreasing sets with general forms, which are positively invariant with respect to the interval system dynamics. Due to the limited length of this paper, the comparative study of the two classes of results could not be illustrated through a numerical example. Even though not explicitly discussed above, the results obtained in this paper for continuous-time positive interval systems can be extended *mutatis mutandis*, to the discrete-time case.

References

[1] R. Sznajder, and M. S. Gowda, "Generalization of P_0- and P-properties; Extended vertical and horizontal linear complementarity problems", Linear Algebra and its Applications, vol. 223/224, pp. 695-715, 1995.

[2] Y. Song, M.S. Gowda, and G. Ravindran, On some properties of P-matrix sets, Linear Algebra and its Applications, vol. 290, pp. 237-246, 1999.

[3] M.M. Moldovan, and M.S. Gowda. "On common linear/ quadratic Lyapunov functions for switched linear systems", In Nonlinear Analysis and Variational Problems, P. Pardalos et al. (Eds.), pp. 415-429, Springer, 2010.

[4] O. Mason, and R. Shorten, "On the simultaneous diagonal stability of a pair of positive linear systems," Linear Algebra and Its Applications, vol. 413, nr. 1, pp. 13–23, 2006.

[5] F. Knorn, O. Mason, and R. Shorten, "On copositive linear Lyapunov functions for sets of linear positive systems", Automatica, vol. 45, pp. 1943-1947, 2009.

[6] E. Fornasini, and M.E. Valcher, "Stability and stabilizability criteria for discrete-time positive switched systems", IEEE Trans. Automatic Control, vol. 57, pp. 1208-1221, 2012.

[7] O. Pastravanu, and M.H. Matcovschi, "Max-type copositive Lyapunov functions for switching positive linear systems". Automatica, vol. 50, nr. 12, pp. 3323-3327, 2014.

[8] O. Pastravanu, M.H. Matcovschi, and M. Voicu, "Row and column representatives in qualitative analysis of arbitrary switching positive systems", Rom. Jrnl. Information Science and Technology, vol. 19, pp. 127–136, 2016.

[9] L. Farina, and S. Rinaldi, Positive linear systems. Theory and applications. Wiley, New York, 2000.

[10] W.M. Haddad, V. Chellaboina, and Q. Hui, Nonnegative and Compartmental Dynamical Systems. Princeton University Press, 2010.

[11] E. Kaszkurewicz, and A. Bhaya, Matrix diagonal stability in systems and computation. Boston. Birkhauser, 2000.

[12] O. Pastravanu, and M.H. Matcovschi, "Diagonal stability of interval matrices and applications", Linear Algebra and Its Applications, vol. 433, nr. 8-10, pp. 1646-1658, 2010.

[13] O Pastravanu, and M.H. Matcovschi, "Stability of matrix polytopes with a dominant vertex and implications for system dynamics", Abstract and Applied Analysis, ID 396759, 2013.

[14] D.S. Bernstein, Matrix Mathematics. Theory, Facts, and Formulas, Princeton University Press, 2009.

[15] R.A. Horn and C.R. Johnson, Topics in Matrix Analysis. Cambridge University Press, Cambridge, 1991.

[16] A. Michel, and K. Wang, Qualitative Theory of Dynamical Systems, Marcel Dekker, Inc., 1995.

[17] J. Albrecht, "Minimal norms of nonnegative irreducible matrices", Linear Algebra and its Applications, vol. 249, nr. 1–3, pp. 255-258, 1996.

[18] L. Kolev, and S. Petrakieva, "Assessing the stability of linear time-invariant continuous interval dynamic systems", IEEE Trans. Automatic Control, vol. 50, nr. 3, pp. 393-397, 2005.